

CATEGORIES OF ORBIT TYPES FOR PROPER LIE GROUPOIDS

JACK MORAVA

ABSTRACT. It is widely understood that the quotient space of a topological group action can have a complicated combinatorial structure, indexed somehow by the isotropy groups of the action [3 II §2.8]; but how best to record this structure seems unclear. This sketch defines a database category of orbit types for a proper Lie groupoid (based on recent work [13-15] with roots in the theory of geometric quantization) as an attempt to capture some of this information.

1. INTRODUCTION AND BACKGROUND

A topological groupoid or stack [12]

$$\mathbf{X} := s, t : X_1 \rightrightarrows X_0$$

is **proper** if the map $s \times t : X_1 \rightarrow X_0 \times X_0$ is proper; such an object in the category of smooth manifolds and maps is a **proper Lie groupoid**. The quotient

$$\mathbf{X} \rightarrow \mathfrak{X}$$

of X_0 by the equivalence relation thus defined is a Hausdorff topological space, sometimes called the coarse moduli space of \mathbf{X} .

Examples:

- Orbifolds [5]
- A topological **transformation group**, defined by a group action

$$G \times X \rightarrow X$$

has an associated topological groupoid $[X/G]$ with $X_0 = X$, $X_1 = G \times X$; I'll write X/G for its quotient space. For instance

- **Toric varieties**, eg $G = \mathbb{T}^{n+1}/\mathbb{T} \cong \mathbb{T}^n$ acting on $X = \mathbb{C}P^n$ by

$$(u_0, \dots, u_n) \cdot [z_0 : \dots : z_n] = [u_0 z_0 : \dots : u_n z_n],$$

form a particularly accessible class of examples. Their quotient objects are polytopes: in the case above $X/G \cong \Delta^n$ is a simplex. The faces of the polytope define a stratification [see §4.1 below] of the quotient, with the

Date: 14 February 2014.

1991 *Mathematics Subject Classification.* 03B65, 14K10, 22A22.

interiors of the faces as strata. This defines an interesting poset, or category, associated to the groupoid: in this example it is the category of subsets of $\{0, \dots, n\}$ under inclusion.

An earlier paper [11] attempted to capture the sort of information encoded by the face poset of a toric variety, for more general group actions. The present note uses recent work on proper Lie groupoids [[16], cf also [1]] to propose a more general construction¹.

Acknowledgement I am indebted to the organizers of the September 2013 Barcelona conference on homotopy type theory [2] for inspiration and hospitality, and for the opportunity to pursue these questions. I hope I will not be misunderstood by suggesting that classification problems of the sort considered here have a deep and nontrivial history in philosophy [17].

2. SOME TECHNICAL PRELIMINARIES

2.1 Definition A reasonable space X has a universal map $X \rightarrow \pi_0 X$ to a discrete set, defined by the adjoint to the inclusion of the category of sets into that of topological spaces. The diagram

$$\begin{array}{ccccccc} X : & & X_1 & \xRightarrow{\quad} & X_0 & \cdots \cdots \cdots & \mathfrak{X} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \pi_0 X : & & \pi_0 X_1 & \xRightarrow{\quad} & \pi_0 X_0 & \cdots \cdots \cdots & \pi_0 \mathfrak{X} \end{array}$$

extends π_0 to a functor from topological to discrete groupoids, such that

$$\pi_0[X/G] \cong [\pi_0(X)/\pi_0(G)] ,$$

with $\pi_0(X)/\pi_0(G) \cong \pi_0(X/G)$ for reasonable actions. The natural transformation

$$\pi_0(X \times Y) \rightarrow \pi_0 X \times \pi_0 Y$$

is an isomorphism in such cases.

2.2 Regarding groups as categories with a single object defines a two-category (**Gps**) of groups. The set of homomorphisms from G_0 to G_1 has an action of G_1 by conjugation, defining a groupoid

$$\mathrm{Hom}_{\mathbf{Gps}}(G_0, G_1) := [\mathrm{Hom}(G_0, G_1)/G_1^{\mathrm{conj}}]$$

of morphisms from G_0 to G_1 .

There are many variations on this theme, eg the topological two-category (**Gps_c**) defined by compact groups and continuous homomorphisms. I will

¹See [8, 21] for approaches based on π_1 rather than π_0

write (\mathbf{Gps}^+) (resp. (\mathbf{Gps}_c^+)) for the subcategories with such groups as objects, and spaces $\mathrm{Hom}_c^+(G_0, G_1)$ of continuous **one-to-one** homomorphisms as maps.

The construction which is the identity on objects, and is the functor

$$[\mathrm{Hom}_c(G_0, G_1)/G_1^{\mathrm{conj}}] \rightarrow \mathrm{Hom}_{\pi_0 \mathbf{Gps}}(G_0, G_1) := \pi_0[\mathrm{Hom}_c(G_0, G_1)/G_1^{\mathrm{conj}}]$$

on morphism categories, defines a monoidal two-functor

$$(\mathbf{Gps}_c) \rightarrow (\pi_0 \mathbf{Gps}_c)$$

[and similarly for (\mathbf{Gps}_c^+)].

3. GROUPOIDS OF FIXED-POINTS WITH LEVEL STRUCTURE

3.1 Definition: If \mathbf{X} is a proper topological groupoid, and H is a compact Lie group, let

$$X(H)_0 := \{(x, \phi) \mid x \in X_0, \phi : H \rightarrow \mathrm{Iso}(x) \in \mathbf{Gps}_c^+\}$$

and let $X(H)_1$ be the set of commutative diagrams of the form

$$\begin{array}{ccc} H & \xrightarrow{\phi'} & \mathrm{Iso}(x') \\ \vdots \gamma & & \downarrow g\text{-conj} \\ H & \xrightarrow{\phi} & \mathrm{Iso}(x) \end{array}$$

(with $g : x' \rightarrow x \in X_1$). The resulting proper topological groupoid $\mathbf{X}(H)$ is a model for the subgroupoid of \mathbf{X} defined by points fixed by a group isomorphic to H . There is a forgetful morphism $\mathbf{X}(H) \rightarrow \mathbf{X}$, but it can't be expected to be the inclusion of a subgroupoid.

Proposition: $H \mapsto \mathbf{X}(H)$ defines a (two-)functor $\mathbf{X}(\bullet)$ from (\mathbf{Gps}_c^+) to the two-category (\mathbf{Gpoids}_c) of proper topological groupoids.

Proof: First of all, if $\alpha : H_0 \longrightarrow H_1 \in (\mathbf{Gps}_c^+)$ then

$$\begin{array}{ccccc} H_0 & \xrightarrow{\alpha} & H_1 & \xrightarrow{\phi'_1} & \mathrm{Iso}(x') \\ \vdots \gamma^\alpha & & & & \downarrow g\text{-conj} \\ H_0 & \longrightarrow & H_1 & \xrightarrow{\phi_1} & \mathrm{Iso}(x) \end{array}$$

defines a functor

$$\alpha_{H_1}^{H_0} : \mathbf{X}(H_1) \rightarrow \mathbf{X}(H_0).$$

Moreover, if $\alpha : H \rightarrow H$ is an inner automorphism of H (ie α is conjugation by $a \in H$) then there is a natural equivalence

$$\alpha_H^H \cong \mathbf{1}_{\mathbf{X}(H)}$$

defined by the commutative diagram

$$\begin{array}{ccccc} H & \xrightarrow{\alpha} & H & \xrightarrow{\phi} & \text{Iso}(x) \\ \vdots \downarrow & & & & \downarrow \phi(a^{-1}) \\ H & \xrightarrow{1_H} & H & \xrightarrow{\phi} & \text{Iso}(x) . \end{array}$$

□

3.2 Claim: For any X as above, there is a commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & (\text{Gpoids}_{c*}) \\ \downarrow \text{Iso} & & \downarrow \\ (\text{Gps}_c) & \xrightarrow{X[\bullet]} & (\text{Gpoids}_c) \end{array}$$

(with the category of pointed proper groupoids in the upper right corner, and the forgetful map to the category of proper groupoids along the right-hand edge). The left-hand vertical map sends $x \in X_0$ to its isotropy group, and the top horizontal map sends x to $X(\text{Iso}(x))$, with x as distinguished point.

Corollary The universal property of a fiber product defines a continuous functor

$$X \rightarrow \Phi_0(X)$$

to the category defined by the pullback

$$\begin{array}{ccccc} \Phi_0(X) & \dashrightarrow & (\text{Gpoids}_{c*}) & \xrightarrow{\pi_0} & (\text{Gpoids})_* \\ \vdots \downarrow & & \downarrow & & \downarrow \\ (\text{Gps}_c) & \xrightarrow{X[\bullet]} & (\text{Gpoids}_c) & \xrightarrow{\pi_0} & (\text{Gpoids}) \end{array}$$

(where the two right vertical arrows are the obvious forgetful functors). □

It's natural to think of $\Phi_0(X)$ as a **database** category [11, 18]. However I don't know how to characterize Φ_0 by some universal property (such as being an adjoint).

3.3 Example: A proper transformation group $[X/G]$ defines a functor

$$G > H \mapsto X^H = \{x \in X \mid \text{Iso}(x) \subset H\}$$

from the topological category $(G - \text{Orb})$ [with closed subgroups of G as objects, and

$$\text{Mor}_{G-\text{Orb}}(H_0, H_1) = \text{Maps}_G(G/H_0, G/H_1) = \{g \in G \mid gH_0g^{-1} \subset H_1\} / H_1^{\text{conj}}$$

as morphism objects [3 I §10], to spaces.

This extends to a functor

$$S^0[X^\bullet] : (G - \text{Spaces}) \ni X \mapsto S^0[X^H] \in \text{Func}(G - \text{Orb}, S^0 - \text{Mod})$$

which provides a model [6 V §9, 10, 20] for the G -equivariant stable category in terms of sheaves of spectra (ie S^0 -modules) over $(G - \text{Orb})$.

The commutative diagram

$$\begin{array}{ccc} [X/G] & \dashrightarrow & (\text{Sets})_* \\ \downarrow \text{Iso} & & \downarrow \\ (G - \text{Orb}) & \xrightarrow{\pi_0 X[*]} & (\text{Sets}) \end{array}$$

defines a functor from $[X/G]$ to a fiber product category $\Phi_0[X/G]$ (with objects, pairs consisting of a subgroup H of G , and a component of X^H), analogous to the construction in the previous paragraph [11 §2.2]. The sheaf $S^\infty X^\bullet$ of spectra pulls back to a sheaf of spectra over $\Phi_0(X)$.

One might hope for an unstable version of this construction, applicable in the theory of ∞ -categories (cf eg [7 §5.5.6.18]); but because it depends on a presentation of $[X/G]$ as a global quotient, it does not seem to be homotopy-invariant.

4. PROPER **Lie** GROUPOIDS, AFTER PFLAUM *et al*

4.1 A stratification \mathcal{S} of a (paracompact, second countable) topological space X assigns to each $x \in X$, the germ of a closed subset \mathcal{S}_x (containing x) of X . With suitably defined morphisms [9 §1.8, 16 §1], stratified spaces form a category. A stratification defines a locally finite partition

$$X = \coprod_{S \in \Sigma(\mathcal{S})} X_S$$

of X into locally closed subsets (called its strata), such that if $x \in X_S$ then \mathcal{S}_x is the associated set germ.

Very interesting recent work of M. Pflaum *et al* [building on earlier work of Weinstein and Zung ([22]; cf also [13]) shows that

Theorem [16 Theorem 5.3, Cor 5.4] The quotient space \mathfrak{X} of a proper Lie groupoid \mathbf{X} has a canonical **Whitney** stratification. The associated decomposition of X_0 into locally closed submanifolds

$$X_{0(H)} = \{x \in X_0 \mid \text{Iso}(x) \cong H\}$$

is indexed [16 Theorem 5.7] by (isomorphism classes of) compact Lie groups H .

4.2 Definition The **orbit** groupoid $\mathcal{O}(x) \subset \mathbf{X}$ of $x \in X_0$

$$\mathcal{O}_0(x) = \{y \in X_0 \mid \exists g : y \rightarrow x \in X_1\}$$

$$\mathcal{O}_1(x) = \{g \in X_1 \mid s(g), t(g) \in \mathcal{O}_0(x)\}$$

reduces, in the case of a transformation groupoid $[X/G]$, to the groupoid

$$[(G/\text{Iso}(x))/G] \equiv [*/\text{Iso}(x)] .$$

A **slice** at $x \in X_0$ is (very roughly [13 §3.3-4, 3.8-9]) the germ of an $\text{Iso}(x)$ -invariant submanifold of X_0 containing x , transverse to $\mathcal{O}_0(x)$; for a transformation group it is something like the image of an exponential map

$$[\mathcal{N}_x/\text{Iso}(x)] \equiv [(\mathcal{N}_x \times_{\text{Iso}(x)} G)/G] \rightarrow [X/G]$$

(where $\mathcal{N}_x \in (\text{Iso}(x) - \text{Mod})$ is the linear representation

$$0 \rightarrow T_x G \rightarrow T_x X_0 \rightarrow \mathcal{N}_x \rightarrow 0 .$$

defining the normal bundle to the orbit of x).

Theorem [16 §3.11] There is an (essentially unique) slice at every object of a proper Lie groupoid \mathbf{X} ; the corresponding set germs define the canonical stratification [16 §5.4] of \mathbf{X} .

Definition The **normal orbit type** of $x \in X_0$ is the equivalence class of its normal $\text{Iso}(x)$ -representation \mathcal{N}_x . More precisely, $x_0 \sim x_1$ if there are isomorphisms

$$\phi : \text{Iso}(x_0) \rightarrow \text{Iso}(x_1), \quad \Phi : \mathcal{N}_{x_0} \rightarrow \phi^*(\mathcal{N}_{x_1})$$

of groups and representations. The connected components $\nu \in \pi_0(X_0)$ of the normal orbit types of \mathbf{X} are [16 §5.7] the strata of the canonical partition of X_0 .

The **condition of the frontier** [16 Prop 5.15] asserts that if $\nu' \cap \bar{\nu} \neq \emptyset$ then $\bar{\nu} \supset \nu'$. This implies the existence of a partial order ($\nu > \nu'$) on the set $\Sigma(\mathbf{X})$ of connected components of normal orbit types for \mathbf{X} , which can thus be regarded as the objects of a category [4 II §2.8]. Thus

$$\overline{X_{0(K)}} = \coprod_{\pi_0(X_{0(K)}) \ni \nu > \nu'} \nu' ,$$

This gives us some control of the functor Φ_0 on proper Lie groupoids:

4.3 Proposition For a proper Lie groupoid \mathbf{X} , we have isomorphisms

$$\bigcup_{H < K} X_{0(K)} \times \text{Hom}_c^+(H, K) \xrightarrow{\cong} X(H)_0$$

$$\bigcup_{H < K, x \in X_{0(K)}} \mathcal{O}_1(x) \times K \xrightarrow{\cong} X(H)_1$$

and consequently

$$\bigcup_{H < K} \mathfrak{X}_K \times \mathrm{Hom}_c^+(H, K)/K^{\mathrm{conj}} \xrightarrow{\cong} \mathfrak{X}(H)$$

(where $\mathfrak{X}_K \subset \mathfrak{X}$ is the space of orbits with isotropy group isomorphic to K).

4.4 Closing remarks

i) When $\mathsf{X} = [X/G]$ this all simplifies a little. In particular, since

$$X^H = \bigcup_{H < K < G} X_K ,$$

$\Phi_0[X/G]$ is essentially just the quotient of $\Phi_0[X/G]$ which collapses the morphism spaces $\mathrm{Hom}_c^+(K, H)/K^{\mathrm{conj}}$.

ii) The subspaces $X_{0(K)}$ are disjoint unions of strata ν indexed by slice representations

$$K \rightarrow \mathrm{Aut}(\mathcal{N}_\nu) .$$

The resulting family of vector spaces over $\mathsf{X}(H)$ pulls back to a fibered category

$$\mathcal{N}(\mathsf{X}) \rightarrow \Phi_0(\mathsf{X}) .$$

This seems to provide a natural repository for Noether's theorem (which associates conserved quantities to elements of the Lie algebra of symmetries of states of a physical system) [11 §4.1].

iii) I don't know how generally one can associate a stratification to a topological groupoid. There are many interesting examples, coming from locally compact groupoids (eg the Thom-Boardman theory of singularities of smooth maps [18]), or from infinite-dimensional examples (Ebin's category of Riemannian metrics up to diffeomorphism, Vassiliev's finite-type invariants of immersions, ...), where a more general theory would be very interesting. The existence and good behavior of slices seem to be an essential requirement for such a theory.

REFERENCES

1. P Cartier, Groupoïdes de Lie et leurs algébroïdes, Sem. Bourbaki 987 (2007-8), Astérisque 326 (2009) 165 - 196
2. A Collective, **Homotopy type theory** . . . , <http://homotopytypetheory.org/book/>
3. T tom Dieck, **Transformation groups**, de Gruyter Studies 8 (1987)
4. JJ Duistermaat, JA Kolk, **Lie groups**, Universitext. Springer (2000)
5. D Gepner, A Henriques, Homotopy theory of orbispaces, [arXiv:math/07019](https://arxiv.org/abs/math/07019)
6. LG Lewis, JP May, M Steinberger, JE McClure, **Equivariant stable homotopy theory**, Springer LNM 1213 (1986)
7. J Lurie, **Higher topos theory**, Annals of Math Studies 170 (2009)
8. W Lück, **Transformation groups and algebraic K-theory**, Springer LNM 1408 (1989)
9. J Mather, Stratifications and mappings, in **Dynamical systems** 195 - 232, ed MM Peixoto, Academic Press (1973)
10. H Miller, The Burnside bicategory of groupoids, [arXiv:1208.2360](https://arxiv.org/abs/1208.2360)
11. J Morava, Theories of anything, [arXiv:1202.0684](https://arxiv.org/abs/1202.0684)
12. B Noohi, Homotopy types of topological stacks, [arXiv:0808.3799](https://arxiv.org/abs/0808.3799)
13. R Palais, On the existence of slices for actions of non- compact Lie groups, Ann. Math. 73 (1961) 295 - 323
14. M Pflaum, **Analytic and geometric study of stratified spaces**, Springer LNM 1768 (2001)
15. —, Smooth structures on stratified spaces, in **Quantization of singular symplectic quotients** 231 - 258, Prog Math (Birkhäuser) 2001
16. —, H Posthuma, X. Tang, Geometry of orbit spaces of proper Lie groupoids, [arXiv:1101.0180](https://arxiv.org/abs/1101.0180)
17. G Scholem, **On the Kabbalah and its symbolism**, Schocken (1996), re <http://en.wikipedia.org/wiki/Sephirot>
18. DI Spivak, R Wisnesky, On The relational foundations Of functorial data migration, [arXiv:1212.5303](https://arxiv.org/abs/1212.5303)
19. R Thom, Singularities of differentiable mapping (notes by H Levine), in **Proceedings of Liverpool Singularities I** 1-89, Springer LNM 1920
20. S Schwede, **Global homotopy theory**, www.math.uni-bonn.de/~schwede/global.pdf
21. J Woolf, The fundamental category of a stratified space, [arXiv:0811.2580](https://arxiv.org/abs/0811.2580)
22. NT Zung, Proper groupoids and momentum maps . . . , Ann Sci École Norm Sup 39 (2006) 841 - 869

THE JOHNS HOPKINS UNIVERSITY, BALTIMORE, MARYLAND 21218

E-mail address: jack@math.jhu.edu